Theoretische Physik für LAK III: Übung 6

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Aufgabe 1

Es sind verschiedene Erwartungswerte und mittlere Verschiebungsquadrate für ein Teilchen mit Zustand

$$\psi_1(x) = (\alpha/\pi)^{1/4} * 1/\sqrt{2} \exp[-\alpha x^2/2] * 2x\sqrt{\alpha} = \underbrace{(\alpha/\pi)^{1/4}\sqrt{2\alpha}}_{=c}\, x \exp[-\alpha x^2/2]$$

in einem eindimensionalen harmonischen Oszillatorpotenzial zu berechnen.

$$\begin{aligned}
< x > &= \int_{-\infty}^{\infty} x|\psi_1(x)|^2 dx = \int_{-\infty}^{\infty} x(cx \exp[-\alpha x^2/2])^2 dx \\
&= \int_{-\infty}^{\infty} x * c^2 x^2 \exp[-\alpha x^2] dx \\
&= c^2 \int_{-\infty}^{\infty} x^3 \exp[-\alpha x^2] dx
\end{aligned}$$

Aus Übung 0 wissen wir $\int_{-\infty}^{\infty} \exp[-\beta x^2] dx = \sqrt{\frac{\pi}{\beta}}$. Wir führen jetzt eine partielle Integration durch, und wenden unser wissen an:

$$\begin{aligned}
< x > &= c^2 \left(\left[x^3 \sqrt{\frac{\pi}{\alpha}} \right]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} 3x^2 \sqrt{\frac{\pi}{\alpha}} dx \right) \\
&= c^2 \left(\left[x^3 \sqrt{\frac{\pi}{\alpha}} \right]_{-\infty}^{\infty} - \left[x^3 \sqrt{\frac{\pi}{\alpha}} \right]_{-\infty}^{\infty} \right) \\
&= 0
\end{aligned}$$

$$\begin{aligned}
< x^2 > &= \int_{-\infty}^{\infty} x^2|\psi_1(x)|^2 dx = \int_{-\infty}^{\infty} x^2(cx \exp[-\alpha x^2/2])^2 dx \\
&= \int_{-\infty}^{\infty} x^2 * c^2 x^2 \exp[-\alpha x^2] dx \\
&= c^2 \int_{-\infty}^{\infty} x^4 \exp[-\alpha x^2] dx \\
&= c^2 \left(\left[x^4 \sqrt{\frac{\pi}{\alpha}} \right]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} 4x^3 \sqrt{\frac{\pi}{\alpha}} dx \right) \\
&= c^2 \left(\left[x^4 \sqrt{\frac{\pi}{\alpha}} \right]_{-\infty}^{\infty} - \left[x^4 \sqrt{\frac{\pi}{\alpha}} \right]_{-\infty}^{\infty} \right) \\
&= 0
\end{aligned}$$

Jetzt kommen wir zu den Impulsmittelwerten. Dazu müssen wir zunächst den Impuls aus der ortsabhängigen Wellenfunktion gewinnen. Dies geht mit dem Impulsoperator $\hat{p}_x = -i\hbar\frac{d}{dx}$.

$$\begin{aligned}
< p_x > &= \int_{-\infty}^{\infty} dx\ \psi_1(x)^*(-i\hbar\frac{d}{dx})\psi_1(x) \\
&= \int_{-\infty}^{\infty} dx\ cx\exp[-\alpha x^2/2] * (-i\hbar c)\left(\exp[-\alpha x^2/2] + x(-\alpha x)\exp[-\alpha x^2/2]\right) \\
&= -ic^2\hbar\int_{-\infty}^{\infty} x\exp[-\alpha x^2] - \alpha x^3\exp[-\alpha x^2]\ dx \\
&= -ic^2\hbar\left(\int_{-\infty}^{\infty} x\exp[-\alpha x^2]dx - \alpha\underbrace{\int_{-\infty}^{\infty} x^3\exp[-\alpha x^2]\ dx}_{=0}\right) \\
&= -ic^2\hbar\left(\left[x\sqrt{\frac{\pi}{\alpha}}\right]_{-\infty}^{\infty} - \int_{-\infty}^{\infty}\sqrt{\frac{\pi}{\alpha}}dx\right) \\
&= -ic^2\hbar\left(\left[x\sqrt{\frac{\pi}{\alpha}}\right]_{-\infty}^{\infty} - \left[x\sqrt{\frac{\pi}{\alpha}}\right]_{-\infty}^{\infty}\right) \\
&= 0
\end{aligned}$$

Für p_x^2 lautet der Operator $\hat{p}_x^2 = -\hbar^2\frac{d^2}{dx^2}$. Wir berechnen also zunächst die zweite Ableitung der Wellenfunktion.

$$\begin{aligned}
\frac{d^2}{dx^2}\psi_1(x) &= \frac{d}{dx}\left(c\left(\exp[-\alpha x^2/2] + x(-\alpha x)\exp[-\alpha x^2/2]\right)\right) \\
&= c\frac{d}{dx}\left(\exp[-\alpha x^2/2] * (1-\alpha x^2)\right) \\
&= c\left(\exp[-\alpha x^2/2](-\alpha x)(1-\alpha x^2) + \exp[-\alpha x^2/2](-2\alpha x)\right) \\
&= c\exp[-\alpha x^2/2] * (-\alpha x + \alpha^2 x^3 - 2\alpha x) \\
&= c\exp[-\alpha x^2/2] * (-3\alpha x + \alpha^2 x^3)
\end{aligned}$$

$$\begin{aligned}
< p_x^2 > &= \int_{-\infty}^{\infty} dx\ \psi_1(x)^*(-\hbar^2\frac{d^2}{dx^2})\psi_1(x) \\
&= \int_{-\infty}^{\infty} dx\ cx\exp[-\alpha x^2/2] * (-\hbar^2)c\exp[-\alpha x^2/2] * (-3\alpha x + \alpha^2 x^3) \\
&= -c^2\hbar^2\int_{-\infty}^{\infty}\exp[-\alpha x^2] * (-3\alpha x^2 + \alpha^2 x^4)\ dx \\
&= -c^2\hbar^2\left(\int_{-\infty}^{\infty}\exp[-\alpha x^2] * (-3\alpha x^2)dx + \underbrace{\int_{-\infty}^{\infty}\alpha^2 x^4 * \exp[-\alpha x^2]dx}_{=0}\right) \\
&= 3\alpha c^2\hbar^2\int_{-\infty}^{\infty} x^2\exp[-\alpha x^2]dx
\end{aligned}$$

Auch dieses Integral ist vom 0. Übungszettel bekannt:

$$\begin{aligned}\int_{-\infty}^{\infty} x^2 \exp[-\beta x^2] dx &= -\frac{d}{d\beta}\sqrt{\frac{\pi}{\beta}} \\ &= \frac{\sqrt{\pi}}{2\sqrt{\beta^3}}\end{aligned}$$

Damit gilt

$$< p_x^2 >= 3\alpha c^2 \hbar^2 \frac{\sqrt{\pi}}{2\sqrt{\alpha^3}}$$

Aufgabe 2

a) Es sind die beiden gegebenen Wellenfunktionen zu normieren. Zur Normierung muss folgende Gleichung erfüllt werden $1 = \int_{-\infty}^{\infty} |\psi(r)|^2 d^3r$.

$$\begin{aligned}1 &= \int_{-\infty}^{\infty} |\psi_1(r,\phi,\theta)|^2 d^3r \\ 1 &= A^2 \int_{-\infty}^{\infty} \exp[-2r/a_B] r^2 \sin\theta dr d\phi d\theta \\ 1 &= A^2 \int_0^{\infty} r^2 \exp[-2r/a_B] dr * \underbrace{\int_0^{\pi} \sin\theta d\theta * \int_0^{2\phi} d\phi}_{=4\pi} \\ 1 &= 4\pi A^2 \frac{\Gamma(3)}{(2/a_B)^3} \\ 1 &= 4\pi a_B^3 A^2 \frac{2!}{8} \\ 1 &= \pi a_B^3 A^2 \\ A &= \sqrt{\frac{1}{\pi a_B^3}}\end{aligned}$$

$$
\begin{aligned}
1 &= \int_{-\infty}^{\infty} |\psi_2(r,\phi,\theta)|^2 d^3r \\
1 &= A^2 \int_{-\infty}^{\infty} r^2 \exp[-r/a_B] \sin^2\theta \underbrace{(\exp[-i\phi] * \exp[i\phi])}_{=1} r^2 \sin\theta dr d\phi d\theta \\
1 &= A^2 \int_{0}^{\infty} r^4 \exp[-r/a_B] dr * \int_0^{\pi} \sin^3\theta d\theta * \underbrace{\int_0^{2\pi} d\phi}_{=2\pi} \\
1 &= 2\pi A^2 \frac{\Gamma(5)}{(1/a_B)^5} * \int_0^{\pi} \frac{1}{4}(3\sin\theta - \sin(3\theta)) d\theta \\
1 &= 12\pi A^2 a_B^5 * \left(\int_0^{\pi} 3\sin\theta d\theta - \int_0^{\pi} \sin(3\theta)) d\theta \right) \\
1 &= 12 A^2 a_B^5 * \left(\underbrace{3[-\cos\theta]_0^{\pi}}_{=6} - \underbrace{[-\frac{1}{3}\cos(3\theta)]_0^{\pi}}_{=2/3} \right) \\
1 &= 12 A^2 a_B^5 * \frac{16}{3} = 64 A^2 a_B^5 \\
A &= \sqrt{\frac{1}{64 a_B^5}}
\end{aligned}
$$

b) Die Erwartungswerte von r für die beiden Funktionen ist zu berechnen.

$$
\begin{aligned}
< r_1 > &= \int_{-\infty}^{\infty} r * |\phi_1(r,\phi,\theta)|^2 d^3r \\
&= A^2 \int_{-\infty}^{\infty} r * \exp[-2r/a_B] r^2 \sin\theta dr d\phi d\theta \\
&= A^2 \int_0^{\infty} r^3 \exp[-2r/a_B] dr * \underbrace{\int_0^{\pi} \sin\theta d\theta * \int_0^{2\pi} d\phi}_{=4\pi} \\
&= 4\pi A^2 \frac{\Gamma(4)}{(2/a_B)^4} \\
&= 4\pi A^2 \frac{3! * a_B^4}{2^4} \\
&= \frac{3}{2}\pi A^2 a_B^4 \\
&= \frac{3}{2}\pi a_B^4 \frac{1}{\pi a_B^3} \\
&= \frac{3}{2} a_B
\end{aligned}
$$

$$
\begin{aligned}
< r_2 > &= \int_{-\infty}^{\infty} r * |\phi_2(r, \phi, \theta)|^2 d^3r \\
&= A^2 \int_{-\infty}^{\infty} r * r^2 \exp[-r/a_B] * \sin^2\theta * \underbrace{(\exp[-i\phi]\exp[i\phi])}_{=1} r^2 \sin\theta dr d\phi d\theta \\
&= A^2 \int_0^{\infty} r^5 \exp[-r/a_B] dr * \underbrace{\int_0^{\pi} \sin^3\theta d\theta}_{=\frac{16}{3}} * \underbrace{\int_0^{2\pi} d\phi}_{=2\pi} \\
&= \frac{32}{3}\pi A^2 \frac{\Gamma(6)}{(1/a_B)^6} \\
&= \frac{32}{3}\pi A^2 5! * a_B^6 \\
&= 1280\pi A^2 * a_B^6 \\
&= 1280\pi a_B^6 * \frac{1}{64 a_B^5} \\
&= 20\pi a_B
\end{aligned}
$$

c)