Aufgabe 13

Es gilt

$$\bar{q}_i^{\,k} = \frac{1}{4}\left(p_i^{k-1} + p_{i+1}^{k-1} + q_i^{k-1} + p^{k-1}\right) \quad \text{und}$$

$$\begin{aligned}
\bar{p}_i^{\,k} &= \frac{1}{4}\left(\bar{q}_{i-1}^{\,k} + \bar{q}_i^{\,k} + p^{k-1} + p_i^{k-1}\right) \\
&= \frac{1}{4}\left(\frac{1}{4}\left[p_{i-1}^{k-1} + p_i^{k-1} + q_{i-1}^{k-1} + p^{k-1}\right] + \frac{1}{4}\left[p_i^{k-1} + p_{i+1}^{k-1} + q_i^{k-1} + p^{k-1}\right] + p^{k-1} + p_i^{k-1}\right) \\
&= \frac{1}{16}\left(p_{i-1}^{k-1} + 6p_i^{k-1} + p_{i+1}^{k-1} + q_{i-1}^{k-1} + q_i^{k-1} + 6p^{k-1}\right).
\end{aligned}$$

Setze an mit der rechten Seite von (2):

$$(1-n\alpha-n\beta)p^{k-1} + \alpha\sum_{i=1}^{n}\bar{p}_i^{\,k} + \beta\sum_{i=1}^{n}\bar{q}_i^{\,k}$$

$$= (1-n\alpha-n\beta)p^{k-1} + \frac{\alpha}{16}\sum_{i=1}^{n}\left(p_{i-1}^{k-1} + 6p_i^{k-1} + p_{i+1}^{k-1} + q_{i-1}^{k-1} + q_i^{k-1} + 6p^{k-1}\right) + \frac{\beta}{4}\sum_{i=1}^{n}\left(p_i^{k-1} + p_{i+1}^{k-1} + q_i^{k-1} + p^{k-1}\right)$$

{Wir betrachten die Summen modulo n, also kommt jeder Index i auch einmal als $i-1$ und einmal als $i+1$ vor.

$$= (1-n\alpha-n\beta)p^{k-1} + \frac{\alpha}{16}\sum_{i=1}^{n}\left(8p_i^{k-1} + 2q_i^{k-1} + 6p^{k-1}\right) + \frac{\beta}{4}\sum_{i=1}^{n}\left(2p_i^{k-1} + q_i^{k-1} + p^{k-1}\right)$$

{p^{k-1} hängt nicht von i ab, also erzeugt jede Summe das n-Fache der vorh. p^{k-1}.

$$= \underbrace{\left(1-n\alpha-n\beta + \frac{6}{16}n\alpha + \frac{1}{4}n\beta\right)}_{(*)}p^{k-1} + \frac{\alpha}{8}\sum_{i=1}^{n}\left(4p_i^{k-1} + q_i^{k-1}\right) + \frac{\beta}{4}\sum_{i=1}^{n}\left(2p_i^{k-1} + q_i^{k-1}\right)$$

{Setze ein: $\alpha = 4\gamma - 8\delta$, $\beta = -2\gamma + 8\delta$

$$\Rightarrow (*) = 1 - 4n\gamma - 8n\delta + 2n\gamma - 8n\delta + \frac{3}{8}n\cdot 4\gamma - \frac{3}{8}n8\delta - \frac{1}{4}n2\gamma + \frac{1}{4}n8\delta$$

$$= 1 - n\gamma - n\delta$$

$$= (1-n\gamma-n\delta)p^{k-1} + \frac{4\gamma-8\delta}{8}\sum_{i=1}^{n}\left(4p_i^{k-1} + q_i^{k-1}\right) + \frac{8\delta-2\gamma}{4}\sum_{i=1}^{n}\left(2p_i^{k-1} + q_i^{k-1}\right)$$

$$= (1-n\gamma-n\delta)p^{k-1} + \frac{1}{2}\gamma\sum_{i=1}^{n}\left(4p_i^{k-1} + q_i^{k-1}\right) - \delta\sum_{i=1}^{n}\left(4p_i^{k-1} + q_i^{k-1}\right) + 2\delta\sum_{i=1}^{n}\left(2p_i^{k-1} + q_i^{k-1}\right) - \frac{1}{2}\gamma\sum_{i=1}^{n}\left(2p_i^{k-1} + q_i^{k-1}\right)$$

{Trenne p_i^{k-1} & q_i^{k-1}

$$\begin{aligned}
&= (1-n\gamma-n\delta)p^{k-1} + 2\gamma\sum_{i=1}^{n}p_i^{k-1} - \gamma\sum_{i=1}^{n}p_i^{k-1} \cancel{- 4\delta\sum_{i=1}^{n}p_i^{k-1} + 4\delta\sum_{i=1}^{n}p_i^{k-1}} \\
&\quad \cancel{+ \frac{1}{2}\gamma\sum_{i=1}^{n}q_i^{k-1} - \frac{1}{2}\gamma\sum_{i=1}^{n}q_i^{k-1}} - \delta\sum_{i=1}^{n}q_i^{k-1} + 2\delta\sum_{i=1}^{n}q_i^{k-1}
\end{aligned}$$

$$= (1-n\gamma-n\delta)p^{k-1} + \gamma\sum_{i=1}^{n}p_i^{k-1} + \delta\sum_{i=1}^{n}q_i^{k-1} \overset{(1)}{=} p^k \qquad \blacksquare$$